

# On Universal Cycles of Labeled Graphs

Greg Brockman, Bill Kay, and Emma E. Snively

Harvard University, University of South Carolina, Rose-Hulman Institute of Technology

November 1, 2009

## Abstract

A universal cycle is a compact listing of a class of combinatorial objects. In this paper, we prove the existence of universal cycles of classes of labeled graphs, including simple graphs, trees, graphs with  $m$  edges, graphs with loops, graphs with multiple edges (with up to  $m$  duplications of each edge), directed graphs, hypergraphs, and  $k$ -uniform hypergraphs.

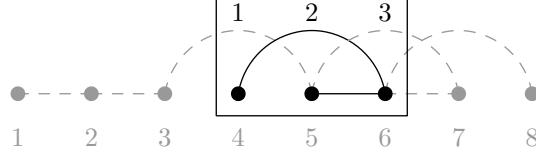
## 1 Introduction

A simple example of a *universal cycle* (U-cycle) is the cyclic string 11101000, which contains every 3-letter word on a binary alphabet precisely once. We obtain these words by taking substrings of length 3; it is useful to imagine that we are looking at the string through a “window” of length 3, and we shift the window to transition from one word to the next, allowing the window to wrap if necessary.

Universal cycles have been shown to exist for words of any length and for any alphabet size. (For the special case of a binary alphabet, such strings are also known as *de Bruijn cycles*). The concept easily lends itself to extension, and universal cycles for permutations, partitions, and certain classes of functions are well-studied in the literature (see Chung, Diaconis, Graham [1] for an overview of previous work in the field). In all cases, the distinguishing feature of a universal cycle is that by shifting a window through a cyclic string (or in some generalizations, an array), all objects in a given class are represented precisely once.

In this paper we generalize the notion of universal cycles. In particular, we show that these cycles exist for certain classes of labeled graphs. In order to define a universal cycle of graphs, we must first extend the notion of a “window.”

**Definition 1.1.** Given a labeled graph  $G$  with vertex set  $V(G) = \{v_0, v_1, \dots, v_{n-1}\}$  with vertices labeled by the rule  $v_j \mapsto j$  and an integer  $0 \leq k \leq n-1$ , define a  $k$ -window of  $G$  to be the subgraph of  $G$  induced by the vertex set  $V = \{v_i, v_{i+1}, \dots, v_{i+k-1}\}$  for some  $i$ , where vertex subscripts are reduced modulo  $n$  as appropriate, and vertices are relabeled such that  $v_i \mapsto 0, v_{i+1} \mapsto 1, \dots, v_{i+k-1} \mapsto k-1$ . For each value of  $i$  such that  $0 \leq i \leq n-1$ , we denote the corresponding  $i^{\text{th}}$   $k$ -window of  $G$  as  $W_{G,k}(i)$ . If  $G$  is clear from context, we abbreviate our window as  $W_k(i)$ .



**Figure 1** — A length 3 window of an 8 vertex graph.

**Definition 1.2.** Given  $\mathcal{F}$ , a family of labeled graphs on  $k$  vertices, a *graph universal cycle* (*GU-cycle*) of  $\mathcal{F}$ , is a labeled graph  $G$  such that the sequence of  $k$ -windows of  $G$  contains each graph in  $\mathcal{F}$  precisely once. That is,  $\{W_k(i) | 0 \leq i \leq n-1\} = \mathcal{F}$ , and  $W_k(i) = W_k(j) \Rightarrow i = j$ . (Note that the vertex set of the  $k$ -windows and the elements of  $\mathcal{F}$  may be different, however, we will set two labeled graphs equal if they differ only by a bijection between their vertex sets.)

**Example 1.3.** Note that the full 8 vertex graph in Figure 1 is a U-cycle of simple labeled graphs (graphs without loops or multiple edges) on 3 vertices.

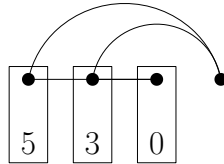
## 2 Universal cycles of simple labeled graphs

We begin our investigation by considering only simple graphs; that is, those without loops or multiple edges. To achieve a form more suited for discourse, we will represent our graphs as strings of integers. The following definition makes this precise.

**Definition 2.1.** Consider any simple labeled graph  $G$  on  $k$  vertices. The *encoded form* of  $G$  is a listing of  $k-1$  integers  $a_0 a_1 \dots a_{k-2}$  chosen as follows:

For each  $0 \leq i \leq k-2$ , set the base 2 representation of  $a_i$  to  $b_{k-1} b_{k-2} \dots b_{i+1}$ , where  $b_m = 1$  if and only if the vertex with label  $i$  is adjacent to the vertex  $m$  in  $G$ .

The  $a_i$  are the *entries* of  $G$ 's encoded form.



**Figure 2** — A graph and its encoded form, 530

The following lemma establishes a bijection between simple labeled graphs and their encoded forms.

**Lemma 2.2.** *Every simple labeled graph has precisely one encoded form, whose entries  $z_i$  satisfy  $0 \leq z_i < 2^{k-i-1}$ . Conversely, let  $A = a_0 a_1 \dots a_{k-2}$  be a string of  $k$  integers such that  $0 \leq a_i < 2^{k-i-1}$ . Then  $A$  is the encoded form of precisely one simple labeled graph on  $k$  vertices.*

*Proof.* Let  $G$  be a simple labeled graph on  $k$  vertices with encoded form  $a_0a_1 \dots a_{k-2}$ . By Definition 2.1, we see that each  $a_i$  has a fixed binary representation. Furthermore, each  $a_i$  is at most  $\underbrace{111 \dots 1}_{k-i-1 \text{ 1s}_2} = 2^{k-i-1} - 1$ . Thus,  $0 \leq a_i < 2^{k-i-1}$ .

We calculate the total number of strings of integers  $a_0a_1 \dots a_{k-2}$  such that  $0 \leq a_i < 2^{k-i-1}$  for each  $i$  as

$$\begin{aligned} \prod_{i=0}^{k-2} 2^{k-i-1} &= \prod_{i=1}^{k-1} 2^i \\ &= 2^{1+\dots+(k-1)} \\ &= 2^{\binom{k}{2}} \end{aligned}$$

This is precisely the number of simple labeled graphs on  $k$  vertices. Since we have established that each such graph has precisely one encoded form, it follows that each such string is the encoded form of exactly one graph.  $\square$

Notice that if a graph  $G$  on  $n$  vertices has encoding  $a_0a_1 \dots a_{n-1}$ , then any  $k$ -vertex subgraph having consecutive vertices labeled  $i, i+1, \dots, i+k-1$  in  $G$  will have encoded form of  $a_i \pmod{2^{k-1}} a_{i+1} \pmod{2^{k-2}} \dots a_{i+k-2} \pmod{2^1}$  (where subscripts and labels are reduced modulo  $n$ ). Thus, in a sense, the string  $a_i a_{i+1} \dots a_{i+k-1}$  is equivalent to an encoding with each entry reduced modulo an appropriate power of two; this gives motivation for defining a U-cycle of encodings.

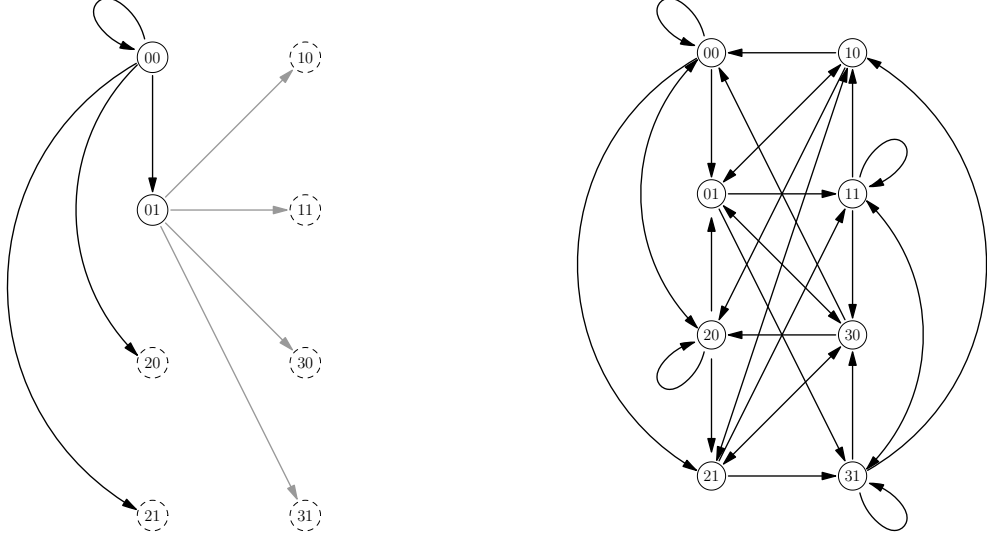
**Definition 2.3.** Given two strings of  $k$  integers,  $A = a_0a_1 \dots a_{k-1}$  and  $B = b_0b_1 \dots b_{k-1}$ , we say that  $A$  and  $B$  are *encoding equivalent* if  $a_i \equiv b_i \pmod{2^{k-i-1}}$  for each integer  $0 \leq i \leq k-1$ . Clearly encoding equivalence is an equivalence relation. Let the canonical representative of each equivalence class be the member  $C = c_0c_1 \dots c_{k-1}$  with  $0 \leq c_i < 2^{k-i-1}$ . Notice that  $C$  is the only member of  $[C]$  which is itself the encoding of a graph. If  $A$  is a string, we let  $C(A)$  denote the graph encoded by the canonical representative from  $A$ 's equivalence class.

**Definition 2.4.** Given a string of integers  $A = a_0a_1 \dots a_{n-1}$  and a positive integer  $k \leq n-1$ , define a  $k$ -window of  $A$  to be a length- $k$  substring of the form  $n_i n_{i+1} \dots n_{i+k-1}$ , where subscripts are reduced modulo  $k$  as appropriate. A *U-cycle of encodings of length  $k$*  is a string of positive integers such that the sequence of  $k$ -windows of the string contains exactly one member from each equivalence class of encodings of length  $k$ .

**Lemma 2.5.** *A U-cycle of encodings of length  $k$  exists if and only if there exists a U-cycle of simple labeled graphs on  $k+1$  vertices.*

*Proof.* Suppose that  $N = n_0n_1 \dots n_{l-1}$  is a U-cycle of encodings of length  $k$ . Then let  $G$  be the graph  $C(N)$ . Notice that  $i^{\text{th}}$   $k$ -window of  $N$  encodes the  $i^{\text{th}}$   $k+1$ -window of  $G$ . Since the collection of  $k$ -windows of  $N$  contains precisely one member from each equivalence class of encodings, we see that the collection of  $k+1$ -windows of  $G$  will contain each possible graph on  $k+1$  vertices exactly once.

Similarly, let  $G$  be a U-cycle of graphs on  $k+1$  vertices. Let  $N = n_0n_1 \dots n_{l-1}$  be its encoding. Since taking every  $k+1$ -window of  $G$  yields every  $k+1$  vertex graph precisely once, we see that taking every  $k$ -window of  $N$  will yield exactly one element from each equivalence class of encodings.  $\square$



**Figure 3** — (Left) A partial sketch of the transition graph of encodings of simple graphs on 3 vertices, and (right) the full transition graph. We provide the left figure for clarity.

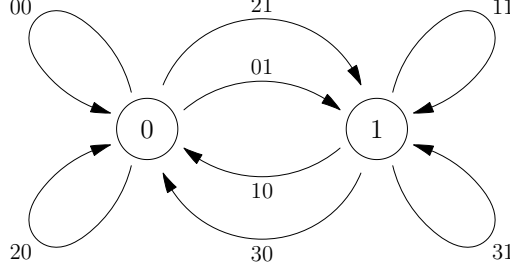
We now prove that U-cycles of simple labeled graphs exist for all  $k \geq 0$ ,  $k \neq 2$ . We employ two common notions from the study of U-cycles: the transition graph and arc digraph. The transition graph  $T$  of a family  $\mathcal{F}$  of combinatorial objects is a directed graph with vertex set  $V(T) = \mathcal{F}$ . Two vertices  $A, B$  are adjacent in  $T$  if and only if  $B$  can follow  $A$  in one window shift of a U-cycle. If  $\mathcal{F}$  is a family of graphs on  $k$  vertices, this means that the subgraph induced by the nodes labeled  $1, 2, \dots, k-1$  in  $A$  is equal to that induced by the nodes  $0, 1, \dots, k-2$  in  $B$ . It should be clear that a U-cycle of  $\mathcal{F}$  corresponds to a Hamiltonian circuit in  $T$  (a directed path passing through every node exactly once).

Unfortunately, finding Hamiltonian circuits in graphs is an NP-hard problem; however, in our case the problem can be further reduced. Let  $D$  be the graph with  $E(D) = \mathcal{F}$  such that two edges  $A, B$  in  $D$  are incident if and only if  $B$  can follow  $A$  in a U-cycle (note that  $V(D)$  is arbitrary). Call  $D$  the *arc digraph* of  $\mathcal{F}$ . Now finding a U-cycle of  $\mathcal{F}$  is equivalent to finding an Eulerian circuit in  $D$  (a directed cycle passing through every edge exactly once); such circuits are easy to detect. For convenience, we often choose the vertices in  $V(D)$  to be labeled with the “overlap” between adjacent edges.

**Lemma 2.6.** *The arc digraph  $D$  of encodings of graphs on  $k+1$  vertices,  $k \geq 2$ , has the following properties:*

1. *For each  $\mathcal{X} \in V(D)$ , the in-degree of  $\mathcal{X}$  equals the out-degree of  $\mathcal{X}$ .*
2. *The graph  $D$  is strongly connected (there is a directed path from  $\mathcal{X}$  to  $\mathcal{Y}$  for any  $\mathcal{X} \neq \mathcal{Y} \in V(D)$ ).*

*Proof.* Fix  $k$ . We begin by constructing the transition graph of equivalence classes of encodings of length  $k$ . The vertices of this graph are our equivalence classes. (We provide an example of such a graph in Figure 3.)



**Figure 4** — The arc digraph for equivalence classes of encodings of length 2 (equivalently, the arc digraph of simple labeled graphs on 3 vertices).

Given two encodings with representatives  $A = a_0a_1 \dots a_{k-1}$ ,  $B = b_0b_1 \dots b_{k-1}$ , we draw an edge from  $A$  to  $B$  if and only if  $B$  can follow  $A$  in a U-cycle. This occurs precisely when we can find a member  $A' = a'_0a'_1 \dots a'_{k-1}$  of  $A$ 's equivalence class and a member  $B' = b'_0b'_1 \dots b'_{k-1}$  of  $B$ 's equivalence class such that  $a'_1 = b'_0, \dots, a'_{k-1} = b'_{k-2}$ . This is equivalent to  $a_i \equiv b_{i-1} \pmod{2^{k-i}}$  for  $1 \leq i \leq k-1$ .

Consider now the arc digraph corresponding to this transition graph. Its edge set will be the vertex set of our transition graph. We will choose the label of each edge to be the canonical representative of that edge's equivalence class. Thus, we may label our graph's vertices as strings of the form  $\alpha_0\alpha_1 \dots \alpha_{k-2}$ , where  $0 \leq \alpha_i < 2^{k-i-1}$ . (Alternatively, we may think about each  $\alpha_i$  as being a residue modulo  $2^{k-i-1}$ .) See Figure 4 for an example.

Let  $\mathcal{X}, \mathcal{Y} \in V(D)$ . Suppose that  $A = a_0a_1 \dots a_{k-1}$  is an edge of this arc digraph that points from  $\mathcal{X}$  to  $\mathcal{Y}$ . Let  $x_i \equiv a_i \pmod{2^{k-i-1}}$  such that  $0 \leq x_i < 2^{k-i-1}$ ; we then have that  $\mathcal{X} = x_0x_1 \dots x_{k-2}$  and  $\mathcal{Y} = x_1x_2 \dots x_{k-1}$ .

**Proof of 1:** Consider any  $\mathcal{X} \in V(D)$ . Let  $\mathcal{X} = x_0x_1 \dots x_{k-2}$ . Then we see that an edge  $A = a_0a_1 \dots a_{k-1}$  has tail  $\mathcal{X}$  if and only if  $x_i \equiv a_i \pmod{2^{k-i-1}}$  for each  $0 \leq i \leq k-2$ . Now for each  $0 \leq i \leq k-2$ , we have that  $0 \leq a_i < 2^{k-i}$  and  $0 \leq x_i < 2^{k-i-1}$ . Thus the possible values for  $a_i$  are  $x_i, x_i + 2^{k-i-1}$ . Hence there are 2 choices for each  $a_i$ , and there are  $2^{k-1}$  possible edges  $A$ . Thus  $\mathcal{X}$  has out-degree  $2^{k-1}$ .

Similarly, an edge  $A = a_0a_1 \dots a_{k-1}$  has head  $\mathcal{X}$  if and only if  $x_i \equiv a_{i+1} \pmod{2^{k-i-1}}$  for each  $0 \leq i \leq k-2$ . For each  $0 \leq i \leq k-2$ , we then have that  $0 \leq x_i < 2^{k-i-1}$ , and it follows that there is only one choice for  $a_{i+1}$ . However,  $a_0$  can now be chosen arbitrarily, so long as  $0 \leq a_0 < 2^{k-1}$ . Hence there are  $2^{k-1}$  possible edges  $A$ , and  $\mathcal{X}$  has in-degree  $2^{k-1}$ , as desired.

**Proof of 2:** Consider any two vertices of  $D$ ,  $\mathcal{X} = x_0x_1 \dots x_{k-2}$  and  $\mathcal{Y} = y_0y_1 \dots y_{k-2}$ . Define  $y_{i,j}$  as the residue of  $y_i \pmod{j}$ , reduced to the interval  $[0, j)$ . We show that the following sequence of vertices defines a path in  $D$  from  $\mathcal{X}$  to  $\mathcal{Y}$ :



**Figure 5** — An illustration of a GU-cycle using a modified window for  $k = 2$ . The left window is the complete graph while the right window is the empty graph. Note that an edge is considered to be in a window only if it is not “cut” by the window.

$$\begin{aligned}
\mathcal{X}_0 &= x_0 x_1 \dots x_{k-2} = \mathcal{X} \\
\mathcal{X}_1 &= x_1 \dots x_{k-2} y_{0,2^1} \\
\mathcal{X}_2 &= x_2 \dots x_{k-2} y_{0,2^2} y_{1,2^1} \\
&\vdots \\
\mathcal{X}_i &= x_i \dots x_{k-2} y_{0,2^i} \dots y_{i-1,2^1} \\
&\vdots \\
\mathcal{X}_{k-2} &= x_{k-2} y_{0,2^{k-2}} \dots y_{k-3,2^1} \\
\mathcal{X}_{k-1} &= y_{0,2^{k-1}} \dots y_{k-2,2^1} = \mathcal{Y}
\end{aligned}$$

Note that each  $\mathcal{X}_i = x_i \dots x_{k-2} y_{0,2^i} \dots y_{i-1,2^1}$  is a valid vertex label, since the size of its entries is appropriately bounded. Also, we can easily check that for each  $0 \leq i \leq k-2$ , there is an edge from  $\mathcal{X}_i$  to  $\mathcal{X}_{i+1}$ .  $\square$

**Theorem 2.7.** *For each  $k \geq 0$ ,  $k \neq 2$ , there exists a universal cycle of simple labeled graphs on  $k$  nodes.*

*Proof.* When  $k = 0$  or  $k = 1$  the result is trivial. For  $k \geq 3$ , Lemma 2.6 implies that the arc digraph of equivalence classes of encodings of length  $k-1$  has an Eulerian cycle, and consequently there exists a U-cycle of equivalence classes of encodings. Then applying Lemma 2.5, we are done.  $\square$

Note that for  $k = 2$  we can modify our definition of a window in order to recognize 2 distinct windows on 2 vertices, as shown in Figure 5. As a final note to the section, notice that in addition to showing existence, our results provide quick algorithms for constructing the relevant GU-cycles.

### 3 General Strategies

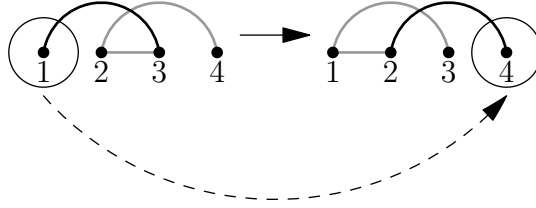
Our approach to showing the existence of GU-cycles in Section 2 was convenient for rigor; however, we feel that working with encodings is cumbersome. In the remainder of this text, we will thus avoid the use of encodings in favor of pictorial manipulations. It should be noted, however, that encodings can be adapted to the situations we are going to consider.

Throughout this section, we suppose that all graphs in a given family have  $k$  vertices for some fixed  $k$ . Since our results will equally well apply to hypergraphs, we will consider hypergraphs to be a class of graphs.

**Definition 3.1.** Let  $\text{rot}(X)$  be the *rotation class* of  $X$ , or the set of labeled graphs that differ from  $X$  only by a cyclic rotation of vertex labels.

**Lemma 3.2.** *Let  $\mathcal{F}$  be a family of labeled graphs (including non-simple graphs or even hypergraphs) such that if  $X \in \mathcal{F}$ , then  $\text{rot}(X) \subseteq \mathcal{F}$ . Then in the arc digraph of  $\mathcal{F}$ , for every vertex  $V$ , the in-degree of  $V$  equals the out-degree of  $V$ .*

*Proof.* Let  $V$  be a vertex of the arc digraph of  $\mathcal{F}$ . Let the set of edges coming in to  $V$  be denoted by  $I(V)$ , and let the set of edges leaving  $V$  be denoted  $O(V)$ . We provide a bijection  $f : I(V) \rightarrow O(V)$ , thus proving our lemma. Let  $I$  be an edge coming into  $V$  (recall that edges in our arc digraph are elements of  $\mathcal{F}$ ). If  $I$  has  $k$  nodes, define  $f(I)$  as the graph obtained by cyclically relabeling  $I$  as follows:  $0 \mapsto k-1, 1 \mapsto 0, 2 \mapsto 1, \dots, k-1 \mapsto k-2$ . Then we see that  $f(I) \in \mathcal{F}$ , since  $f(I)$  is a rotation of  $I$ , and furthermore  $f(I)$  is an edge leaving  $V$ .



**Figure 6** — The cyclic relabeling of the nodes to create an isomorphic graph

Injectivity of  $f$  is clear. Now consider any edge  $J$  leaving  $V$ . Let  $I$  be the graph obtained by cyclically relabeling  $J$  as follows:  $0 \mapsto 1, 1 \mapsto 2, \dots, k-2 \mapsto k-1, k-1 \mapsto 0$ . Again,  $I$  is isomorphic to  $J$ , so  $I \in \mathcal{F}$ . Furthermore,  $I$  is an edge coming into  $V$ . Thus  $J$  has a preimage under  $f$ , and  $f$  is surjective.  $\square$

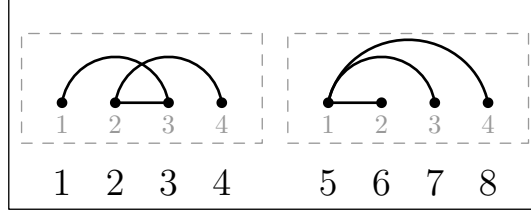
Lemma 3.2 implies that if some class  $\mathcal{F}$  of labeled graphs is closed under rotation, then in order to show that a GU-cycle of  $\mathcal{F}$  exists we need only show that the arc digraph is strongly connected (save for isolated nodes). That is, we must show that given two edges  $I$  and  $J$  in the arc digraph, there exists a directed path in the arc digraph beginning with  $I$  and ending with  $J$ . In terms of GU-cycles, this is equivalent to showing the existence of a graph  $G$  such that  $W_{G,k}(i) = I, W_{G,k}(j) = J$  and  $W_{G,k}(h) \in \mathcal{F}$  for  $i \leq h \leq j$ . Or alternatively, we can picture walking on the arc digraph from  $I$  to  $J$ , taking a series of “moves” from one incident edge to another, always following the directed arrows.

We now apply these ideas to prove the existence of U-cycles of various classes of graphs.

**Theorem 3.3.** *U-cycles exist for the following classes of graphs: graphs with loops, graphs with multiple edges (with up to  $m$  duplications of each edge), directed graphs, hypergraphs, and  $k$ -uniform hypergraphs.*

*Proof.* Take  $\mathcal{F}$  to be any of the above classes of graphs. Pick two graphs  $I$  and  $J$  from  $\mathcal{F}$ . Relabel the nodes of  $J$  by the rule  $i \mapsto k+i$  to create a graph  $G$  such that  $V(G) = V(I) \cup V(J)$  and  $E(G) = E(I) \cup E(J)$  (see Figure 7). The graph  $I$  is the first  $k$ -window of  $G$ , and the graph  $J$  is the  $k^{\text{th}}$   $k$ -window. Further, each  $k$ -window  $W_{G,k}(i)$ ,  $1 \leq i \leq k$ , is a graph in  $\mathcal{F}$ . Thus these  $k$ -windows represent a series of legal edge moves in our arc digraph.  $\square$

The extensions from Theorem 3.3 followed readily because the relevant graph classes were unrestricted; connectedness of the arc digraph was trivial. Notice that our proof also applies



**Figure 7** — Relabeling  $J$  to create a new graph  $G$

to some restricted classes of graphs, such as forests. We now turn our attention to GU-cycles of two types of restricted classes of simple graphs on  $k$  vertices.

**Theorem 3.4.** *Graph universal cycles exist for trees on  $k$  nodes for  $k \geq 3$ .*

*Proof.* Let  $I, J$  be trees. Let  $G$  be the graph obtained by concatenating  $I$  and  $J$  as in the proof of Theorem 3.3. As we read the  $k$ -windows starting from  $I$ , let  $M$  be the first non-tree window that we arrive upon. Let  $v_M$  be vertex of highest label in  $M$ . Since none of  $G$ 's subgraphs contain cycles, we see that there must be one or more components of  $M$  that are not connected to the component of  $v_M$ . Draw edges from  $v_M$  to each of these components; note that the resulting window is now a tree. Furthermore, these edges did not create any cycles in any of  $G$ 's  $k$ -windows, since there are no edges between vertices with label higher than that of  $v_M$  and those with lower label. Also note that still  $W_{G,k}(1) = I, W_{G,k}(k) = J$ . We then iterate this process until we arrive at a graph that gives us a sequence of  $k$ -windows, all of which are trees, starting at  $I$  and ending at  $J$ .  $\square$

**Theorem 3.5.** *Graph universal cycles exist for graphs with precisely  $m$  edges.*

*Proof.* For any graph  $G$ , let  $d(G)$  be the degree sequence of  $G$ . For two graphs  $G, H$  having  $m$  edges, define  $d(G) < d(H)$  if  $d(G)$  comes before  $d(H)$  lexicographically. We show that for any graph  $I$  having exactly  $m$  edges, there is a sequence of moves that takes  $I$  to the (unique) graph  $L$  having  $m$  edges and having least degree sequence of graphs with  $m$  edges. Thereupon, using the bijection we created in Lemma 3.2, for any graph  $J$  having exactly  $m$  edges we can reverse its path to  $L$  to arrive at a path from  $L$  to  $J$ . This will complete our proof.

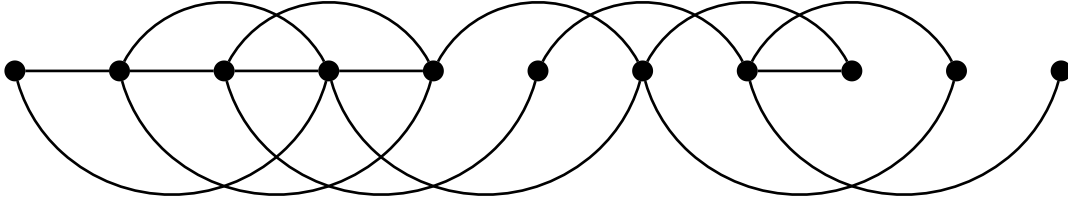
Let  $I$  be a graph with  $m$  edges. If  $I$  has least degree sequence, we are done. Otherwise, let  $d(I) = (d_0, d_1, \dots, d_{k-1})$  and  $d(L) = (L_0, L_1, \dots, L_{k-1})$ . Since  $d(I)$  is not minimal, there must be some  $i$  such that  $d_i > L_i$ . But then there must exist some  $j > i$  such that  $d_j < L_j$ . Now consider a sequence of moves where at each step we rotate  $I$ 's vertices according to the relabeling  $0 \mapsto k-1, 1 \mapsto 0, \dots, k-1 \mapsto k-2$  until the vertex formerly labeled  $i$  attains label 0; let  $i'$  be any vertex adjacent to it. Once more rotate the vertex set, but this time connect the vertex formerly labeled  $i'$  to  $j$  instead of  $i$ . After rotating all of the vertices back to their original positions, we obtain a graph with a smaller degree sequence than  $I$ . By infinite descent, we see that we can eventually move to  $L$  via a sequence of moves all of which are graphs having  $m$  edges, thus completing the proof of the theorem.  $\square$



## 4 Conclusions and Future Directions

In this paper, we have presented a beginning theory of universal cycles of graphs. We have shown the existence of U-cycles of various classes of labeled graphs on  $k$  nodes, including simple graphs, multigraphs, graphs on  $m$  edges, directed graphs, trees, hypergraphs, and  $k$ -uniform hypergraphs.

Our work in this field is far from complete. There exist many other classes of graphs for which there conceivably exist U-cycles. However, perhaps the most obvious gap is results regarding U-cycles of unlabeled graphs. The canonical result would be to prove the existence of U-cycles of isomorphism classes of graphs. In such a U-cycle, no two  $k$ -windows are isomorphic and every isomorphism class is represented as a  $k$ -window. It is easy to find a U-cycle of isomorphism classes of graphs on 3 nodes. It is difficult, but still possible, to find a U-cycle of isomorphism classes of graphs on 4 nodes; one such cycle is exhibited in Figure 8. These results lead us to conjecture the following.



**Figure 8** — A graph universal cycle of isomorphism classes of graphs on 4 nodes.

**Conjecture 4.1.** *For each  $k \geq 3$ , there exists a U-cycle of isomorphism classes of graphs on  $k$  nodes.*

We also note that graph universal cycles have potential in theorem-proving as well, as demonstrated by the following result.

**Definition 4.2.** We say that an integer-valued graph theoretic function  $f$  is *window-Lipschitz* if, for all graphs  $G$  and  $H$  which are one window shift apart in a graph universal cycle,  $|f(G) - f(H)| \leq 1$ .

**Lemma 4.3.** *Let  $U$  be a U-cycle of some family  $\mathcal{F}$  of graphs, and let  $f$  be a window-Lipschitz function defined on these graphs. Then for each integer  $\min_{G \in \mathcal{F}} f(G) < i < \max_{G \in \mathcal{F}} f(G)$  there exist at least two distinct elements of  $G \in \mathcal{F}$  such that  $f(G) = i$ .*

*Proof.* By definition of a Lipschitz function, under a single window shift the value of  $f$  can change by at most 1. Hence during the sequence of window shifts from the graph with minimal  $f$ -value to maximal, every possible value of  $f$  in between the minimum and maximum is attained. Similarly, during the sequence of window shifts from the graph with maximal  $f$ -value to that with minimal, every possible value of  $f$  is again attained. This completes our proof.  $\square$

Some examples of window-Lipschitz functions are chromatic number and largest clique.

## 5 Acknowledgements

This work was done at the East Tennessee State University REU, NSF grant 0552730, under the supervision of Dr. Anant Godbole.

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- [1] P. Diaconis F. Chung and R. Graham, *Universal cycles for combinatorial structures*, Discrete Mathematics **110** (1992), 43–59.

GREG BROCKMAN  
HARVARD UNIVERSITY  
CAMBRIDGE, MA 02138  
UNITED STATES  
gbrockm@fas.harvard.edu

BILL KAY  
UNIVERSITY OF SOUTH CAROLINA  
COLUMBIA, SC 29208  
UNITED STATES  
kayw@mailbox.sc.edu

EMMA E. SNIVELY  
ROSE-HULMAN INSTITUTE OF TECHNOLOGY  
TERRE HAUTE, IN 47803  
UNITED STATES  
snivelee@rose-hulman.edu

# On Universal Cycles of Labeled Graphs

Submitted: Aug 26, 2008; Accepted: XX; Published: XX

Mathematics Subject Classification: 05C30

GREG BROCKMAN  
HARVARD UNIVERSITY  
CAMBRIDGE, MA 02138  
UNITED STATES  
brockman@hcs.harvard.edu

BILL KAY  
UNIVERSITY OF SOUTH CAROLINA  
COLUMBIA, SC 29208  
UNITED STATES  
kayw@mailbox.sc.edu

EMMA E. SNIVELY  
ROSE-HULMAN INSTITUTE OF TECHNOLOGY  
TERRE HAUTE, IN 47803  
UNITED STATES  
snivelee@rose-hulman.edu

## Abstract

A universal cycle is a compact listing of a class of combinatorial objects. In this paper, we prove the existence of universal cycles of classes of labeled graphs, including simple graphs, trees, graphs with  $m$  edges, graphs with loops, graphs with multiple edges (with up to  $m$  duplications of each edge), directed graphs, hypergraphs, and  $k$ -uniform hypergraphs.

## 1 Introduction

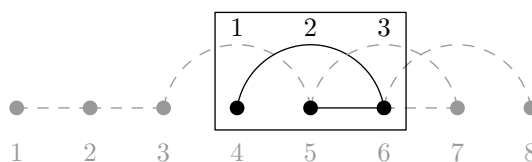
A simple example of a *universal cycle* (U-cycle) is the cyclic string 11101000, which contains every 3-letter word on a binary alphabet precisely once. We obtain these words by taking substrings of length 3; it is useful to imagine that we are looking at the string through a “window” of length 3, and we shift the window to transition from one word to the next, allowing the window to wrap if necessary.

Universal cycles have been shown to exist for words of any length and for any alphabet size. (For the special case of a binary alphabet, such strings are also known as *de Bruijn cycles*). The concept easily lends itself to extension, and universal cycles for permutations, partitions, and certain classes of functions are well-studied in the literature (see Chung, Diaconis, Graham [1] for an overview of previous work in the field). In all cases, the distinguishing feature of a universal cycle is that by shifting a window through a cyclic

string (or in some generalizations, an array), all objects in a given class are represented precisely once.

In this paper we generalize the notion of universal cycles. In particular, we show that these cycles exist for certain classes of labeled graphs. In order to define a universal cycle of graphs, we must first extend the notion of a “window.”

**Definition 1.1.** Given a labeled graph  $G$  having vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$  with vertices labeled by the rule  $v_j \mapsto j$  and an integer  $0 \leq k \leq n$ , define a  $k$ -window of  $G$  to be the subgraph of  $G$  induced by the vertex set  $V = \{v_i, v_{i+1}, \dots, v_{i+k-1}\}$  for some  $i$ , where vertex subscripts are reduced modulo  $n$  as appropriate, and vertices are relabeled such that  $v_i \mapsto 1, v_{i+1} \mapsto 2, \dots, v_{i+k-1} \mapsto k$ . For each value of  $i$  such that  $1 \leq i \leq n$ , we denote the corresponding  $i^{\text{th}}$   $k$ -window of  $G$  as  $W_{G,k}(i)$ . If  $G$  is clear from context, we abbreviate our window as  $W_k(i)$ .



**Figure 1** — A 3-window of an 8 vertex graph.

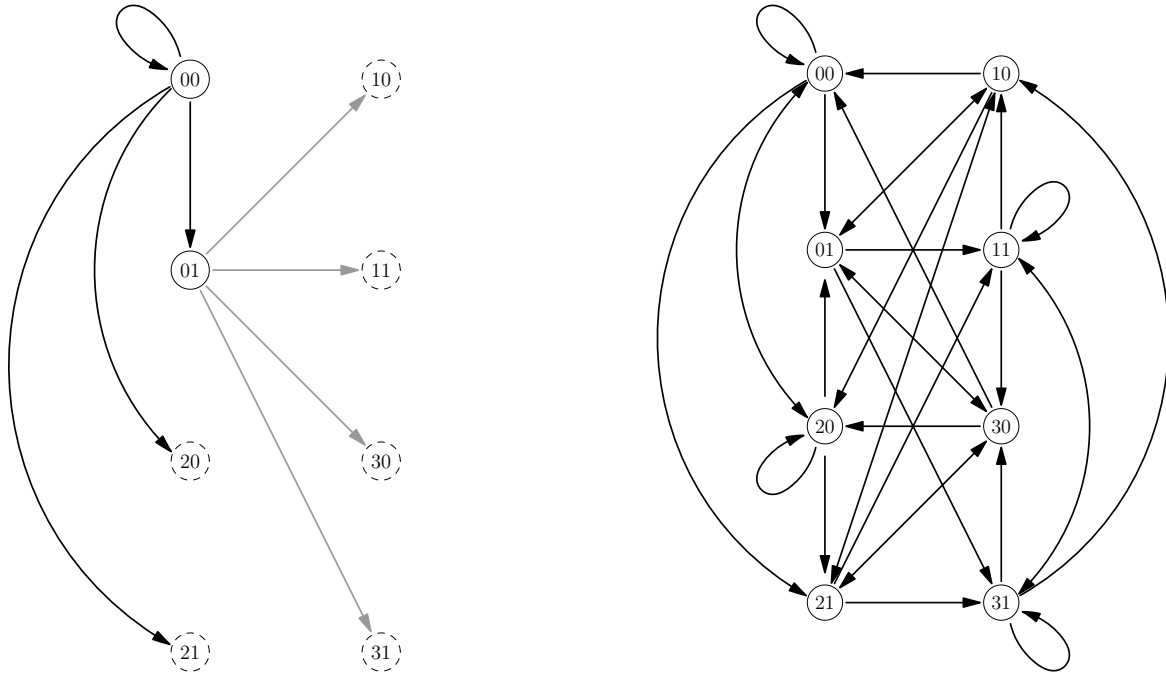
**Definition 1.2.** Given  $\mathcal{F}$ , a family of labeled graphs on  $k$  vertices, a *universal cycle* ( $U$ -cycle) of  $\mathcal{F}$ , is a labeled graph  $G$  such that the sequence of  $k$ -windows of  $G$  contains each graph in  $\mathcal{F}$  precisely once. That is,  $\{W_k(i) | 1 \leq i \leq n\} = \mathcal{F}$ , and  $W_k(i) = W_k(j) \implies i = j$ . (Note that the vertex set of the  $k$ -windows and the elements of  $\mathcal{F}$  may be different, however, we will set two labeled graphs equal if they differ only by a bijection between their vertex sets.)

**Example 1.3.** Note that the full 8 vertex graph in Figure 1 is a U-cycle of simple labeled graphs (graphs without loops or multiple edges) on 3 vertices.

## 2 Universal cycles of simple labeled graphs

We begin our investigation by considering only simple graphs; that is, those without loops or multiple edges. Our result will be that U-cycles of simple labeled graphs on  $k$  vertices exist for all  $k \geq 0$ ,  $k \neq 2$ .

Our proof employs two common notions from the study of U-cycles: the transition graph and arc digraph. The transition graph  $T$  of a family  $\mathcal{F}$  of combinatorial objects is a directed graph with vertex set  $V(T) = \mathcal{F}$ . If  $A, B \in \mathcal{F}$ , there is an edge from  $A$  to  $B$  in  $T$  if and only if  $B$  can follow  $A$  in one window shift of a U-cycle. If  $\mathcal{F}$  is a family of graphs on  $k$  vertices, this means that the subgraph induced by the vertices labeled  $2, 3, \dots, k$  in  $A$  is equal to that induced by the vertices  $1, 2, \dots, k-1$  in  $B$ . It should be clear that a



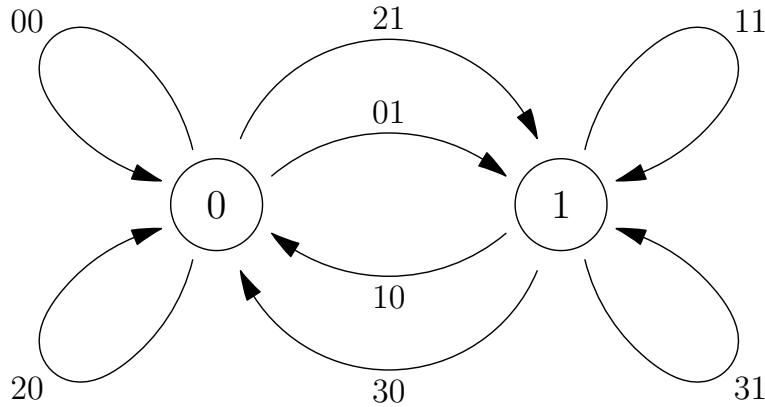
**Figure 2** — (Left) A partial sketch of the transition graph of simple graphs on 3 vertices, and (right) the full transition graph. We provide the left figure for clarity.

U-cycle of  $\mathcal{F}$  corresponds to a Hamiltonian circuit in  $T$  (a directed cycle passing through every vertex exactly once).

Unfortunately, finding Hamiltonian circuits in graphs is an NP-hard problem; however, in our case the problem can be further reduced. Let  $D$  be the graph with  $E(D) = \mathcal{F}$  such that two edges  $A, B$  in  $D$  are consecutive (the head of  $A$  equals the tail of  $B$ ) if and only if  $B$  can follow  $A$  in a U-cycle. Note that  $V(D)$  is arbitrary. Call  $D$  the *arc digraph* of  $\mathcal{F}$ . Now finding a U-cycle of  $\mathcal{F}$  is equivalent to finding an Eulerian circuit in  $D$  (a directed cycle passing through every edge exactly once); such circuits are easy to detect. In particular, a graph has an Eulerian circuit if and only if each of its vertices has equal in-degree and out-degree and the graph is strongly connected (for any two vertices  $x, y$ , there is a directed path from  $x$  to  $y$ ). For convenience, we often choose the vertices in  $V(D)$  to be equal to the “overlap” between consecutive edges. In the following, we construct the transition graph only as a guide for constructing the arc digraph.

**Lemma 2.1.** *The arc digraph  $D$  of simple labeled graphs on  $k$  vertices,  $k \geq 3$ , has the following properties:*

1. *For each  $\mathcal{X} \in V(D)$ , the in-degree of  $\mathcal{X}$  equals the out-degree of  $\mathcal{X}$ .*
2. *The graph  $D$  is strongly connected (there is a directed path from  $\mathcal{X}$  to  $\mathcal{Y}$  for any  $\mathcal{X} \neq \mathcal{Y} \in V(D)$ ).*



**Figure 3** — The arc digraph of the set of simple labeled graphs on 3 vertices.

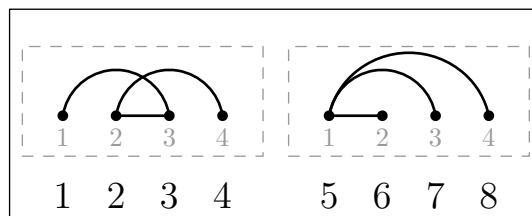
*Proof.* Fix  $k \geq 3$ . Let  $\mathcal{F}$  be the set of simple graphs on  $k$  vertices. We begin by constructing the transition graph of  $\mathcal{F}$ . The vertices of this graph are the elements of  $\mathcal{F}$ . As an example, Figure 2 contains the transition graph for the case  $k = 3$ .

Consider  $A, B \in \mathcal{F}$ . Let  $u$  be the subgraph induced by the vertices labeled  $2, 3, \dots, k$  in  $A$  (with its vertices relabeled to  $1, 2, \dots, k-1$ , preserving order) and let  $v$  be the subgraph induced by the vertices labeled  $1, 2, \dots, k-1$  in  $B$ . We draw an edge from  $A$  to  $B$  if and only if  $B$  can follow  $A$  in a U-cycle, which is equivalent to  $u = v$ . Thus the transition graph has an edge from  $A$  to  $B$  if and only if removing the first vertex from  $A$  yields the same graph as removing the last vertex from  $B$ .

We now construct the arc digraph corresponding to this transition graph. Its edge set will be the vertex set of our transition graph. In accordance with the convention mentioned earlier, we use as its vertex set the set of graphs on  $k-1$  vertices. By the previous paragraph, the head of an edge  $A \in \mathcal{F}$  is the vertex equal to the induced subgraph resulting from removing  $A$ 's first vertex. Similarly its tail is the vertex equal to the induced subgraph resulting from removing its last vertex. See Figure 3 for the arc digraph in the case  $k = 3$ .

**Proof of 1:** Let  $\mathcal{X} \in V(D)$  be a vertex in the arc digraph. Since an edge  $A \in E(D)$  points into  $\mathcal{X}$  if and only if removing the first vertex of  $E(D)$  yields  $\mathcal{X}$ , the in-degree of  $\mathcal{X}$  must equal  $2^{k-1}$ , since the first vertex can arbitrarily be adjacent to or not adjacent to each vertex in  $V(\mathcal{X})$ . Similarly, the out-degree of  $\mathcal{X}$  is also  $2^{k-1}$  since an edge  $B \in E(D)$  points out of  $\mathcal{X}$  precisely when deleting its last vertex yields  $\mathcal{X}$ , and again we have two choices for each vertex in  $V(D)$ .

**Proof of 2:** Consider any two vertices of  $D$ ,  $\mathcal{X}$  and  $\mathcal{Y}$ . Let  $G$  be the (labeled) disjoint union of  $\mathcal{X}$  and  $\mathcal{Y}$ , after incrementing the label on each of  $\mathcal{Y}$ 's vertices by  $k$ , as exemplified in Figure 4. Now consider the sequence of  $k$ -vertex graphs  $W_{G,k}(1), W_{G,k}(2), \dots, W_{G,k}(k+1)$ . Deleting the first vertex of  $W_{G,k}(i)$  yields  $W_{G,k-1}(i+1)$ , as does deleting the last vertex of  $W_{G,k}(i+1)$ . Thus  $W_{G,k}(i)$  and  $W_{G,k}(i+1)$  are consecutive in  $D$ . Furthermore,  $W_{G,k}(1) = \mathcal{X}$  and  $W_{G,k}(k+1) = \mathcal{Y}$ , and hence there is a path in  $D$  from  $\mathcal{X}$  to  $\mathcal{Y}$ , as desired.



**Figure 4** — An example of taking the labeled disjoint union of  $\mathcal{X}$  and  $\mathcal{Y}$ , where each of  $\mathcal{X}$  and  $\mathcal{Y}$  are graphs on four vertices.

□

**Theorem 2.2.** *For each  $k \geq 0$ ,  $k \neq 2$ , there exists a universal cycle of simple labeled graphs on  $k$  vertices.*

*Proof.* When  $k = 0$  or  $k = 1$  the result is trivial. For  $k \geq 3$ , Lemma 2.1 implies that the arc digraph of simple labeled graphs on  $k$  vertices has an Eulerian cycle, and hence a U-cycle of them exists. □

Note that for  $k = 2$  we can modify our definition of a window in order to recognize two distinct windows on two vertices, as shown in Figure 5. Also notice that in addition to showing existence, our results provide quick algorithms for constructing the relevant U-cycles.

### 3 General Strategies

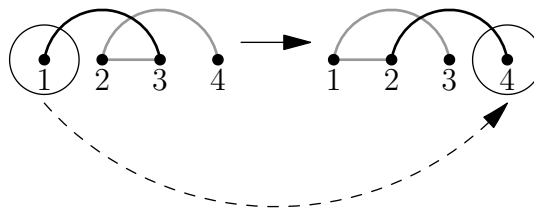
The results of the previous section can be generalized to many classes of graphs, as we show here. Throughout this section, we suppose that all graphs in a given family have  $k$  vertices for some fixed  $k$ . Since our results will equally well apply to hypergraphs, we will consider hypergraphs to be a class of graphs.

**Definition 3.1.** Let  $\text{rot}(X)$  be the *rotation class* of  $X$ , or the set of labeled graphs that differ from  $X$  only by a cyclic rotation of vertex labels.

**Lemma 3.2.** *Let  $\mathcal{F}$  be a family of labeled graphs (possibly including non-simple graphs or even hypergraphs) such that if  $X \in \mathcal{F}$ , then  $\text{rot}(X) \subseteq \mathcal{F}$ . Then in the arc digraph of  $\mathcal{F}$ , for every vertex  $V$ , the in-degree of  $V$  equals the out-degree of  $V$ .*



**Figure 5** — An illustration of a U-cycle using a modified window for  $k = 2$ . The left window is the complete graph while the right window is the empty graph. Note that an edge is considered to be in a window only if it is not “cut” by the window.



**Figure 6** — The cyclic relabeling of the vertices to create an isomorphic graph

*Proof.* Let  $V$  be a vertex of the arc digraph of  $\mathcal{F}$ . Let the set of edges pointing into  $V$  be denoted by  $I(V)$ , and let the set of edges leaving  $V$  be denoted  $O(V)$ . We provide a bijection  $f : I(V) \longrightarrow O(V)$ , thus proving our lemma. Let  $I$  be an edge pointing into  $V$  (recall that edges in our arc digraph are elements of  $\mathcal{F}$ ). If  $I$  has  $k$  vertices, define  $f(I)$  as the graph obtained by cyclically relabeling  $I$  as follows:  $1 \mapsto k, 2 \mapsto 1, 3 \mapsto 2, \dots, k \mapsto k-1$ . Then we see that  $f(I) \in \mathcal{F}$ , since  $f(I)$  is a rotation of  $I$ , and furthermore  $f(I)$  is an edge leaving  $V$ .

Injectivity of  $f$  is clear. Now consider any edge  $J$  leaving  $V$ . Let  $I$  be the graph obtained by cyclically relabeling  $J$  as follows:  $1 \mapsto 2, 2 \mapsto 3, \dots, k-1 \mapsto k, k \mapsto 1$ . Again,  $I$  is isomorphic to  $J$ , so  $I \in \mathcal{F}$ . Furthermore,  $I$  is an edge pointing into  $V$ . Thus  $J$  has a preimage under  $f$ , and  $f$  is surjective.  $\square$

Lemma 3.2 implies that if some class  $\mathcal{F}$  of labeled graphs is closed under rotation, then in order to show that a U-cycle of  $\mathcal{F}$  exists we need only show that the arc digraph is strongly connected (save for isolated vertices). That is, we must show that given two edges  $I$  and  $J$  in the arc digraph, there exists a directed path in the arc digraph beginning with  $I$  and ending with  $J$ . In terms of U-cycles, this is equivalent to showing the existence of a graph  $G$  such that  $W_{G,k}(i) = I, W_{G,k}(j) = J$  and  $W_{G,k}(h) \in \mathcal{F}$  for  $i \leq h \leq j$ . Or alternatively, we can picture walking on the arc digraph from  $I$  to  $J$ , taking a series of “moves” along consecutive edges, always following the directed arrows.

We now apply these ideas to prove the existence of U-cycles of various classes of graphs.

**Theorem 3.3.** *For each  $k \neq 2$ , U-cycles exist for the following classes of graphs on  $k$  vertices: graphs with loops, graphs with multiple edges (with up to  $m$  duplications of each edge), directed graphs, hypergraphs, and  $j$ -uniform hypergraphs.*

*Proof.* The cases  $k = 0, 1$  are trivial. If  $k \geq 3$ , we proceed in analogy to Part 2 of Lemma 2.1. Take  $\mathcal{F}$  to be any of the desired classes of graphs. Pick two graphs  $I$  and  $J$  from  $\mathcal{F}$ . Let  $G$  be the labeled disjoint union of  $I$  and  $J$ . The graph  $I$  is the first  $k$ -window of  $G$ , and the graph  $J$  is the  $(k+1)^{\text{st}}$   $k$ -window. Further, each  $k$ -window  $W_{G,k}(i)$ ,  $1 \leq i \leq k+1$ , is a graph in  $\mathcal{F}$ . Thus these  $k$ -windows represent a series of legal edge moves in our arc digraph.  $\square$

The extensions from Theorem 3.3 followed readily because the relevant graph classes were unrestricted; connectedness of the arc digraph was trivial. Notice that our proof also applies to some restricted classes of graphs, such as forests. We now turn our attention to U-cycles of two types of restricted classes of simple graphs on  $k$  vertices.



**Theorem 3.4.** *U-cycles exist for trees on  $k$  vertices for  $k \geq 3$ .*

*Proof.* Let  $I, J$  be trees. Let  $G$  be the labeled disjoint union of  $I$  and  $J$ . As we read the  $k$ -windows starting from  $I$ , let  $M$  be the first non-tree window that we arrive upon. Define  $v_M$  to be vertex of highest label in  $M$ . Since none of  $G$ 's subgraphs contain cycles, we see that there must be one or more components of  $M$  that are not connected to the component of  $v_M$ . Draw edges from  $v_M$  to each of these components; note that the resulting window is now a tree. Furthermore, these edges did not create any cycles in any of  $G$ 's  $k$ -windows, since there are no edges between vertices with label higher than that of  $v_M$  and those with lower label. Also note that still  $W_{G,k}(1) = I, W_{G,k}(k+1) = J$ . We then iterate this process until we arrive at a graph that gives us a sequence of  $k$ -windows, all of which are trees, starting at  $I$  and ending at  $J$ .  $\square$

**Theorem 3.5.** *U-cycles exist for graphs with precisely  $m$  edges.*

*Proof.* For any graph  $G$ , let  $d(G)$  be the degree sequence of  $G$ . For two graphs  $G, H$  having  $m$  edges, define  $d(G) < d(H)$  if  $d(G)$  comes before  $d(H)$  lexicographically. We show that for any graph  $I$  having exactly  $m$  edges, there is a sequence of moves that takes  $I$  to the (unique) graph  $L$  having  $m$  edges and having least degree sequence of graphs with  $m$  edges. Thereupon, using the bijection we created in Lemma 3.2, for any graph  $J$  having exactly  $m$  edges we can reverse its path to  $L$  to arrive at a path from  $L$  to  $J$ . This will complete our proof.

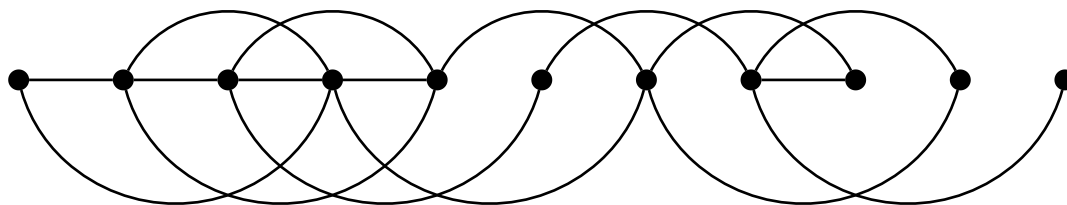
Let  $I$  be a graph with  $m$  edges. If  $I$  has least degree sequence, we are done. Otherwise, let  $d(I) = (d_1, d_2, \dots, d_k)$  and  $d(L) = (L_1, L_2, \dots, L_k)$ . Since  $d(I)$  is not minimal, there must be some  $i$  such that  $d_i > L_i$ . But then there must exist some  $j > i$  such that  $d_j < L_j$ . Now consider a sequence of moves where at each step we rotate  $I$ 's vertices according to the relabeling  $1 \mapsto k, 2 \mapsto 1, \dots, k \mapsto k-1$  until the vertex formerly labeled  $i$  attains label 1; let  $i'$  be any vertex adjacent to it. Rotate the vertex set once more, but this time connect the vertex formerly labeled  $i'$  to  $j$  instead of  $i$ . After rotating all of the vertices back to their original positions, we obtain a graph with a smaller degree sequence than  $I$ . By infinite descent, we see that we can eventually move to  $L$  via a sequence of moves all of which are graphs having  $m$  edges, thus completing the proof of the theorem.  $\square$

## 4 Conclusions and Future Directions

In this paper, we have presented a beginning theory of universal cycles of graphs. We have shown the existence of U-cycles of various classes of labeled graphs on  $k$  vertices, including simple graphs, multigraphs, graphs on  $m$  edges, directed graphs, trees, hypergraphs, and  $k$ -uniform hypergraphs.

Our work in this field is far from complete. There exist many other classes of graphs for which there conceivably exist U-cycles. However, perhaps the most obvious gap is results regarding U-cycles of unlabeled graphs. The canonical result would be to prove the existence of U-cycles of isomorphism classes of graphs. In such a U-cycle, no two  $k$ -windows are isomorphic and every isomorphism class is represented as a  $k$ -window. It

is easy to find a U-cycle of isomorphism classes of graphs on 3 vertices. It is difficult, but still possible, to find a U-cycle of isomorphism classes of graphs on 4 vertices; one such cycle is exhibited in Figure 7. These results lead us to conjecture the following.



**Figure 7** — A U-cycle of isomorphism classes of graphs on 4 vertices.

**Conjecture 4.1.** *For each  $k \neq 2$ , there exists a U-cycle of isomorphism classes of graphs on  $k$  vertices.*

We also note that U-cycles have potential in theorem-proving as well, as demonstrated by the following result.

**Definition 4.2.** We say that an integer-valued graph theoretic function  $f$  is *window-Lipschitz* if, for all graphs  $G$  and  $H$  which are one window shift apart in a U-cycle,  $|f(G) - f(H)| \leq 1$ .

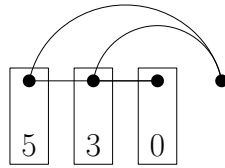
Some examples of window-Lipschitz functions are chromatic number and largest clique.

**Lemma 4.3.** *Let  $U$  be a U-cycle of some family  $\mathcal{F}$  of graphs, and let  $f$  be a window-Lipschitz function defined on these graphs. Then for each integer  $\min_{G \in \mathcal{F}} f(G) < i < \max_{G \in \mathcal{F}} f(G)$  there exist at least two distinct elements of  $G \in \mathcal{F}$  such that  $f(G) = i$ .*

*Proof.* By definition of a Lipschitz function, under a single window shift the value of  $f$  can change by at most 1. Hence during the sequence of window shifts from the graph with minimal  $f$ -value to maximal, every possible value of  $f$  in between the minimum and maximum is attained. Similarly, during the sequence of window shifts from the graph with maximal  $f$ -value to that with minimal, every possible value of  $f$  is again attained. This completes our proof.  $\square$

Finally, we note that it is possible to reduce finding a U-cycle of a set of labeled graphs to finding a U-cycle of an appropriately defined set of equivalence classes of words. For example, let  $\mathcal{F}$  be the set of simple labeled graphs on  $k$  vertices, and let  $\mathcal{G}$  be the set of words of length  $k - 1$  on the alphabet  $\{0, 1, \dots, 2^{k-1} - 1\}$ . Define  $f : \mathcal{G} \rightarrow \mathcal{F}$  such that  $f(x_1 x_2 \dots x_{k-1})$  is the graph where, for  $1 \leq i < j \leq k$ , there is an edge from  $i$  to  $j$  if and only if the  $j$ th bit of  $x_i$  is 1. Now define two words in  $\mathcal{G}$  to be equivalent when their image under  $f$  is equal, and define the bijection  $f'$  to map an equivalence class to the image under  $f$  of any member of that equivalence class. For an example, see Figure 8.

It is not hard to show that a U-cycle of these equivalence classes (a string whose sequence of  $(k - 1)$ -windows contains exactly one representative from each equivalence



**Figure 8** — A graph and an element of its corresponding equivalence class of words, 530.

class) exists precisely when a U-cycle of  $\mathcal{F}$  exists. This reduction allows one to think of U-cycles of graphs in the more traditional context of U-cycles of a restricted class of words. Similar reductions apply to other classes of labeled graphs.

## 5 Acknowledgements

This work was done at the East Tennessee State University REU, NSF grant 0552730, under the supervision of Dr. Anant Godbole.

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- [1] P. Diaconis F. Chung and R. Graham, *Universal cycles for combinatorial structures*, Discrete Mathematics **110** (1992), 43–59.